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2009 J. Phys. A: Math. Theor. 42 135202

(http://iopscience.iop.org/1751-8121/42/13/135202)

IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **42** (2009) 135202 (8pp) doi:10.1088/1751-8113/42/13/135202

A Baker–Campbell–Hausdorff solution by differential equation

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Received 18 August 2008, in final form 11 February 2009
Published 4 March 2009
Online at stacks.iop.org/JPhysA/42/135202

Abstract
We propose a procedure to figure out the Baker–Campbell–Hausdorff (BCH) solution, $\ln e^X e^Y$, when the exponent is a linear combination of the spin operator along a direction and its ladder operators. The procedure converts the manipulation of the BCH formula into that of a differential equation. It is shown that the fixed point of the differential equation leads to the solution we are looking for. We also remark that the validity of the present method is restricted to the case when the solution branch can be determined in the complex plane.

PACS numbers: 02.30.Tb, 02.20.Sv, 02.30.Jr, 02.30.Hq

1. Introduction

The time evolution of a system is usually described by $\partial_t \Psi(t) = \mathcal{L}\Psi(t)$, where $\Psi(t)$ is a status of the system at time t and $\mathcal{L}$ is a time-independent linear operator. Thus $\mathcal{L}$ contains the whole information regarding the dynamics of the system. The state-ket in Hilbert space, the probability density, and the density operator are evolved this way with their own operators $\mathcal{L}$s [1–4]. Liouville's theorem reveals the same description for the evolution of the density of system points in the phase space [5]. When the evolution of Ψ is considered for an infinitesimal duration δt, after linearization, it reads $\Psi(t + \delta t) = (1 + \delta t \mathcal{L})\Psi(t)$. Applying this process repeatedly during the evolution time from 0 to t, it follows that:

$$\Psi(t) = \lim_{N\to\infty} \left(1 + \frac{t}{N}\mathcal{L}\right)^N \Psi(0) \equiv e^{t\mathcal{L}}\Psi(0), \tag{1}$$

where $e^{t\mathcal{L}}$ is called an evolution operator. This is the formal solution of $\partial_t \Psi(t) = \mathcal{L}\Psi(t)$ for the boundary condition at time $t = 0$. Besides, equation (1) reveals

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!} \tag{2}$$

for an operator X, the algebraic structure of which is exactly same as that of the ordinary exponential function for number. In this way $\mathcal{L}$ generates the evolution operator $e^{t\mathcal{L}}$, and is called an evolution generator. Conversely, the logarithm of the evolution operator is proportional to the evolution generator, and thus the essence of the dynamics is encoded in it.

According to equation (1), when $\mathcal{L}_1$ is applied for t_1 and then $\mathcal{L}_2$ for t_2 follows, the evolution reads $\Psi(t_1+t_2) = e^{t_2\mathcal{L}_2}\, e^{t_1\mathcal{L}_1}\Psi(0)$. This is directly extended to $\Psi(t_1+t_2+\cdots+t_n) = e^{t_n\mathcal{L}_n}\cdots e^{t_2\mathcal{L}_2}\, e^{t_1\mathcal{L}_1}\Psi(0)$ when n number of generators are considered. Actually, this kind of series operation frequently occurs in the nuclear magnetic resonance experiment [6, 7] and in other types of coherent spectroscopy [8]. Quantum computing supposing a well-prepared entangled state [4, 9] and the theory regarding squeezing operators [10] are also examples. This is why the required state is usually squeezed through the series operation of a few generators in control. Furthermore, the series operation is also used even in the numerical study, and the associated numerical technique is called an operator-splitting scheme. For example, the Suzuki–Trotter decomposition method [11, 12] is employed in integrating the time-dependent Schrödinger equations in *ab initio* first-principle calculations. A similar method is also used in studying the properties of stochastic systems [13, 14]. The operator-splitting scheme is an alternative when it is hardly possible to realize the whole integrand directly in the conventional way of numerical integration.

In the case of series operation, it is a question what the effective evolution generator ($\mathcal{L}_{\text{eff}}$) is, which can be considered to govern the whole process consistently. As mentioned above, the generator is proportional to the logarithm of the evolution operator. Thus when the proportional constant is set to be 1, the logarithm of the evolution operator may represent $\mathcal{L}_{\text{eff}}$ without any loss of generality[1]. As an example, for the two-step process, it follows that $\mathcal{L}_{\text{eff}} = \ln e^{t_2\mathcal{L}_2}\, e^{t_1\mathcal{L}_1}$. However, this is merely a formal expression of $\mathcal{L}_{\text{eff}}$, symbolically defined, and thus it hardly gives any information regarding the overall dynamics except a few trivial cases like when $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1$. Therefore, it is challenging to obtain the informative expression of $\mathcal{L}_{\text{eff}}$ from the formal one. A classic work on this profound issue is [15] by Wilcox, where various conventional approaches are synthesized. Recently, an approach based on Cayley–Hamilton theory was proposed [16], which is applicable when $\mathcal{L}_i$ and $\mathcal{L}_2$ are the finite dimensional matrices.

In this work, we design a procedure to figure out Z satisfying $e^Z = e^X\, e^Y$, that is, $z = \ln e^X\, e^Y$ when X and Y are respectively the linear combination of a spin operator along a direction and its lowering/raising operators. To assertain the significance of spin operators, this problem has already been studied, more than four decades ago [15, 17]. We here present a different and a rather intuitive procedure to solve it. First, we briefly introduce the Baker–Campbell–Hausdorff (BCH) formula, a systematic manipulation of $\ln e^X\, e^Y$, in section 2. In section 3, for the case of $[X, Y] = X$ or Y, we put forward an idea to convert the present problem into the fixed-point problem of a differential equation. This is then generalized to the above-mentioned case in section 4. Finally, we finish this work in section 5 with some remarks regarding the limited applicability of the present result as well as a summarizing conclusion.

[1] The proportional constant is determined according to the peculiarity of the system under investigation. However, the algebraic complexity, caused by the logarithm, remains the same whatever value is assigned to the constant. In this sense, we only focus on the handling of the logarithm itself.

2. Baker–Campbell–Hausdorff formula

The 'Baker–Campbell–Hausdorff (BCH) formula' is concerned with the logarithm of $\mathrm{e}^X\,\mathrm{e}^Y$, that is, looking for Z such that $\sum_{k=0}^{\infty} Z^k/k! = \mathrm{e}^X\,\mathrm{e}^Y$ holds. In principle, the solution is to be obtained by comparing the two sides of $\sum_{k=0}^{\infty} Z^k/k! = \left(\sum_{k=0}^{\infty} X^k/k!\right)\left(\sum_{k=0}^{\infty} Y^k/k!\right)$, term by term. The most familiar version of the BCH formula is [18]

$$Z = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] \\ + \cdots + k_{w_1,\ldots,w_n}[w_1,[..[w_n,[X,Y]]..]] + \cdots, \tag{3}$$

where $[X,Y] \equiv XY - YX$, w_i stands for either X or Y, and $k_{w_1,\ldots,w_n}$ is a real scalar. However, the useful expression of $k_{w_1,..,w_n}$ is not known yet, and thus equation (3) is hardly useful as far as actual computation is concerned [18]. The importance of this version is because all terms in the series, except the leading $X + Y$, are written in terms of repeated brackets of X and Y.

Another version of the BCH formula worth mentioning is [19]

$$Z = X + \int_0^1 \mathrm{d}t\, g(\mathrm{e}^{\mathrm{ad}_X}\,\mathrm{e}^{t\,\mathrm{ad}_Y})Y, \tag{4}$$

where $g(x) \equiv 1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m(m+1)}(x-1)^m$ and ad_X is a linear map the operation of which is defined by $\mathrm{ad}_X Y = [X,Y]$. One may use equation (4) to obtain the $k_{w_1,\ldots,w_n}$s of equation (3). However, this integral form of BCH formula is still not so useful in practical study because of its highly sophisticated structure. Exceptionally, when $[X,Y] = X$ or Y holds, this version is useful to obtain the simple expression of Z. This will follow in the next section.

3. A simple non-Abelian case and fixed-point method

Equation (4) is able to determine the closed form of Z when

$$[X,Y] = X \text{ or } Y \tag{5}$$

is provided. This algebra has been already well studied due to its significance in physics [17, 15, 20], and it was already known that

$$Z = X + Y + \left(\frac{1}{e-1}\right)[X,Y]. \tag{6}$$

Recently, [14] has applied $x\ln x/(x-1) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m(m+1)}(x-1)^m$ to equation (4) to arrive at the same result. In the following, it will be shown that this Z is also obtainable through the fixed point of a differential equation.

In order to study in a different way, we introduce two differential operators: $X = \hat{P}\rho^2$ and $Y = -\hat{P}\rho$, where $\hat{P} = \partial/\partial\rho$. Note that this holds $[X,Y] = X$ (the other case, $[X,Y] = Y$, will be considered later). When $[X,Y] = X$ is provided, equation (3) leads to $Z = \alpha X + Y$ for a constant α. This is because (i) any term of which w_is represent X, at least once, becomes null in the series of equation (3), (ii) while the others, whose w_is are now always Y, remain proportional to X. Thus it follows that

$$\mathrm{e}^{\hat{P}\rho^2}\,\mathrm{e}^{-\hat{P}\rho} = \mathrm{e}^{\hat{P}(\alpha\rho^2-\rho)}. \tag{7}$$

It is interesting to interpret the exponential operators in equation (7) as the time evolution operators during unit time, respectively. When the leading one, $\mathrm{e}^{\hat{P}\rho^2}$, is interpreted this way, one may consider $\Psi(\rho,t) = \mathrm{e}^{t\hat{P}\rho^2}\Psi(\rho,0)$ as the formal solution of $\partial_t\Psi = \hat{P}\rho^2\Psi$, equivalently,

$$\frac{\partial}{\partial t}\Psi(\rho,t) = -\frac{\partial}{\partial\rho}(-\rho^2\Psi(\rho,t)) \tag{8}$$

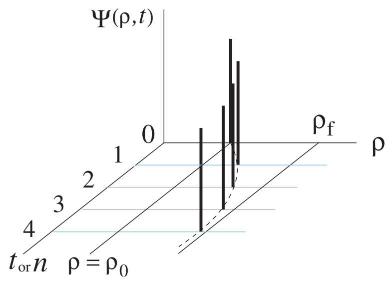

Figure 1. The bold solid lines stand for the delta-peak distribution at discrete times ($n = 0, 1, 2, \ldots$). The peak initially begins at $n = 0$, and evolves according to the map in equation (10). Therein ρ_f is the stable fixed point of the map. The dashed curve on the t–ρ plane is the solution of $\dot{\rho} = -\alpha\rho^2 + \rho$ with ρ_0-initialization, where $\alpha = 1/\rho_\mathrm{f}$. Note that this α makes the solution curve contact with all the delta-peaks, that is, $\rho_n = \rho(n)$. See the text for detail.

with $\Psi(\rho, 0)$-initialization. Herein, one may also interpret $\Psi(\rho, t)$ as a distribution over a spatial coordinate ρ at time t and $-\rho^2$ as the velocity field defined on the plane of (ρ, t). In this case, equation (8) is no more than a continuity equation describing the flow of the distribution whose strip located at ρ has velocity $v(\rho) = -\rho^2$. As a consequence, microscopically, a particle in the strip at time t is driven by

$$\dot{\rho}(t) = -\rho^2(t) \tag{9}$$

as long as its dynamics is deterministic[2].

When a particle is represented by a delta-peak distribution, equations (8) and (9) say that the evolution of the delta-peak is completely determined by the solution of equation (9). We note that the easily solvable ordinary differential equation in equation (9) completely traces out the trajectory of any delta-peak distribution. For the other exponential operators in equation (7), $\mathrm{e}^{-\hat{P}\rho}$ and $\mathrm{e}^{\hat{P}(\alpha\rho^2-\rho)}$, one similarly finds $\dot{\rho}(t) = \rho(t)$ and $\dot{\rho}(t) = -\alpha\rho^2(t) + \rho(t)$, respectively. These are also easily solvable, and thus reveal useful information in the same manner.

When a delta-peak at $\rho = \rho_0$ is evolved by the lhs of equation (7), $\mathrm{e}^{\hat{P}\rho^2}\,\mathrm{e}^{-\hat{P}\rho}$, its position first follows $\dot{\rho} = \rho$ and then $\dot{\rho} = -\rho^2$. Herein, each step lasts for unit time. Since the solutions are $\rho(t) = \rho_0\,\mathrm{e}^t$ and $\rho(t) = \rho_0/(1 + \rho_0 t)$, respectively, the initial peak moves to $\rho = \rho_0 e/(1+\rho_0 e)$. This is the combination of the two solutions where the unit-time evolution is considered in each case. When this is repeatedly applied, the positions of the peaks form a map to follow

$$\rho_{n+1} = \frac{\rho_n e}{1 + \rho_n e}, \tag{10}$$

where ρ_n is the location of the peak at nth step. This map has two fixed points 0 and $(e-1)/e$, and the former is unstable and the other is stable. Figure 1 shows the evolution of the delta-peak distribution according to ρ_n.

We next pay attention to the fact that the same fixed points [0 and $(e-1)/e$] should be also realized by the rhs of equation (7), $\mathrm{e}^{\hat{P}(\alpha\rho^2-\rho)}$. This requirement is due to the equality in equation (7). The microscopic dynamics is $\dot{\rho} = -\alpha\rho^2 + \rho$ this time, and then one may take $\alpha = e/(e-1)$ to fulfil the requirement. Figure 1 also shows the solution to this differential

[2] The correspondence between equations (8) and (9) can be regarded as the deterministic counterpart of that between Fokker–Planck and Langevin equations considered in the presence of stochasticity. See [2] for details.

equation. Note that the stability of the each fixed point is also preserved. Thus if $[X, Y] = X$ is provided, it follows that

$$\mathrm{e}^X\,\mathrm{e}^Y = \mathrm{e}^{\frac{e}{e-1}X+Y}, \tag{11}$$

which corresponds to one case of equation (6).

For the other case of $[X, Y] = Y$, one may introduce $X = \hat{P}\rho$ and $Y = \hat{P}\rho^2$, which hold $[X, Y] = Y$. This time, in equation (3), the leading $X + Y$ and the terms where all w_is are X contribute to Z. Then it is immediate that $\mathrm{e}^{\hat{P}\rho}\,\mathrm{e}^{\hat{P}\rho^2} = \mathrm{e}^{\hat{P}(\rho+\beta\rho^2)}$. Following the same procedure explained above, one finds that 0 and $(1 - e)/e$ should be the fixed points of the $\dot{\rho} = -\rho - \beta\rho^2$. This directly leads to $\beta = e/(e - 1)$. Thus for $[X, Y] = Y$, one finds that

$$\mathrm{e}^X\,\mathrm{e}^Y = \mathrm{e}^{X+\frac{e}{e-1}Y}. \tag{12}$$

As a consequence, the results in equations (11) and (12) show that equation (6) is also obtainable by the method presented in this section. We below name it the fixed-point method.

4. Generalization to split three-dimensional simple algebra

The 'split three-dimensional simple algebra' [15, 17] is defined by $\mathcal{B} = \{B_1, B_2, B_3\}$ satisfying

$$[B_1, B_2] = 2B_2, \qquad [B_1, B_3] = -2B_3, \qquad [B_2, B_3] = B_1. \tag{13}$$

This algebra is of wide interest in physics because it is actually equivalent to the fundamental commutation relations of the angular momentum operators [1]. It is also realized in the isotopic spin formalism [21], where the algebra of two uncoupled harmonic oscillators is devised to show the commutator relations in equation (13). A set of $\{\mathrm{i}J_-, \mathrm{i}J_+, 2J_z\}$ is such an example when J_z is the angular momentum operator along the z-axis and J_+ and J_- are the angular momentum raising and lowering operators, respectively. We below consider the case where X and Y are the linear combination of B_is, that is, $X = x \cdot B = \sum_{i=1}^{3} x_i B_i$ and $Y = y \cdot B = \sum_{i=1}^{3} y_i B_i$. For these X and Y, due to the closure of $\mathcal{B}$, equation (3) leads to

$$\mathrm{e}^{z\cdot B} = \mathrm{e}^{x\cdot B}\,\mathrm{e}^{y\cdot B}, \tag{14}$$

where $z = (z_1, z_2, z_3)$ is now under investigation.

In order to generalize the fixed-point method, let us take

$$B_1 = 2\hat{P}\rho, \qquad B_2 = -\hat{P}\rho^2, \qquad B_3 = \hat{P}. \tag{15}$$

Note that these B_is exactly hold the condition of equation (13). Besides, when

$$f_x(\rho) \equiv x_2\rho^2 - 2x_1\rho - x_3 \tag{16}$$

is introduced ($f_y(\rho)$ and $f_z(\rho)$ are also considered this way below), one may write $\mathrm{e}^{x\cdot B} = \mathrm{e}^{-\hat{P}f_x(\rho)}$. Remark that this can be understood as the unit-time evolution operator for an arbitrary distribution $\Psi(\rho, t)$ from which evolution follows:

$$\partial_t\Psi = -\hat{P}f_x(\rho)\Psi. \tag{17}$$

Now if $f_x(\rho)$ is interpreted as the velocity field at ρ, equation (17) is no more than a continuity equation, where ρ is treated as the spatial coordinate. Thus when a delta-peak distribution of $\Psi(\rho, 0) = \delta(\rho - \rho_0)$ is used for an initial condition, it remains peak and moves while its trajectory is subject to

$$\dot{\rho} = f_x(\rho). \tag{18}$$

Herein, we use the fact that equation (17) has no terms to bring about any diffusion originated from stochasticity. Then one finds $\Psi(\rho, t) = \delta(\rho - \rho_x(\rho_0; t))$, where $\rho_x(\rho_0; t)$ stands for the solution of equation (18) at time t with ρ_0-initialization.

The same argument is also possible for $e^{y\cdot B}$ when $f_y(\rho)$ is introduced. Thus when the rhs of equation (14), $e^{x\cdot B}\,e^{y\cdot B}$, is applied to $\delta(\rho-\rho_0)$, one first considers $e^{y\cdot B}\delta(\rho-\rho_0)$, and next the result is applied to $e^{x\cdot B}$. Then it follows that

$$e^{x\cdot B}\,e^{y\cdot B}\delta(\rho-\rho_0)=\delta(\rho-\rho_x(\rho_y(\rho_0;1);1)). \tag{19}$$

In continuing, the other side of equation (14), $e^{z\cdot B}$, is also treated in the similar manner to find

$$e^{z\cdot B}\delta(\rho-\rho_0)=\delta(\rho-\rho_z(\rho_0:1)). \tag{20}$$

As a consequence, equating equation (19) with equation (20), one finds

$$\rho_z(\rho_0;1)=\rho_x(\rho_y(\rho_0;1);1). \tag{21}$$

In order to go forward, the concrete expression of $\rho_x(\rho_0;t)$ is required. Following elementary calculus, one straightforwardly obtains that

$$\rho_x(\rho_0;t)=\frac{(x_+-x_-\,e^{tx_2\Delta x})\rho_0+x_+x_-(e^{tx_2\Delta x}-1)}{(1-e^{tx_2\Delta x})\rho_0+(x_+\,e^{tx_2\Delta x}-x_-)}, \tag{22}$$

where $x_\pm=x_1\pm\sqrt{x_1^2+x_2x_3}$ and $\Delta x\equiv x_+-x_-$. Thus when this result is applied to $\rho_x(\rho_y(\rho_0;1);1)$, it is direct that $\rho_x(\rho_y(\rho_0;1);1)=(a\rho_0+b)/(c\rho_0+d)$, where

$$\begin{cases} a=(x_+-x_-\,e^{x_2\Delta x})(y_+-y_-\,e^{y_2\Delta y})+x_+x_-\,(e^{x_2\Delta x}-1)(1-e^{y_2\Delta y}),\\ b=(x_+-x_-\,e^{x_2\Delta x})y_+y_-(e^{y_2\Delta y}-1)+x_+x_-(e^{x_2\Delta x}-1)(y_+\,e^{y_2\Delta y}-y_-),\\ c=(1-e^{x_2\Delta x})(y_+-y_-\,e^{y_2\Delta y})+(x_+\,e^{x_2\Delta x}-x_-)(1-e^{y_2\Delta y}),\\ d=(1-e^{x_2\Delta x})y_+y_-(e^{y_2\Delta y}-1)+(x_+\,e^{x_2\Delta x}-x_-)(y_+\,e^{y_2\Delta y}-y_-). \end{cases} \tag{23}$$

Herein $y_\pm=y_1\pm\sqrt{y_1^2+y_2y_3}$ and $\Delta y=y_+-y_-$.

When the iterated operation of $e^{x\cdot B}e^{y\cdot B}$ on ρ_0 is considered, equation (22) leads to a map of

$$\rho_{n+1}=\frac{a\rho_n+b}{c\rho_n+d}, \tag{24}$$

where ρ_n is the result after nth operation. Then the fixed point of this map, ρ_{fix}, satisfies $c\rho_{\mathrm{fix}}^2-(a-d)\rho_{\mathrm{fix}}-b=0$. Due to equation (14), the related fixed points should be also realized by the iterated operation of $e^{z\cdot B}$. That is, the fixed points have to be encoded in $\dot\rho=f_z(\rho)$. Consequently, one can write

$$f_z(\rho)=z_2\left(\rho^2-\frac{a-d}{c}\rho-\frac{b}{c}\right). \tag{25}$$

Thus it follows that

$$z=(z_1,z_2,z_3)=\left(\frac{a-d}{2c}z_2,\,z_2,\,\frac{b}{c}z_2\right), \tag{26}$$

which can be completely fixed up if z_2 is determined.

One may rewrite equation (25) as $f_z(\rho)=z_2(\rho-z_+)(\rho-z_-)$ for $z_\pm=\left(\frac{a-d}{c}\pm\sqrt{\left(\frac{a-d}{c}\right)^2+\frac{4b}{c}}\right)\Big/2$. Then $\rho_z(\rho_0;1)$ is directly revealed by equation (22), and this also leads to a map of

$$\rho_{n+1}=\frac{(z_+-z_-\,e^{z_2\Delta z})\rho_n+z_+z_-(e^{z_2\Delta z}-1)}{(1-e^{z_2\Delta z})\rho_n+(z_+\,e^{z_2\Delta z}-z_-)}, \tag{27}$$

where $\Delta z\equiv z_+-z_-$. After the comparison between equations (24) and (27), one finds

$$\frac{a}{c}=\frac{z_+-z_-\,e^{z_2\Delta z}}{1-e^{z_2\Delta z}}. \tag{28}$$

Finally, one obtains

$$z_2 = \frac{1}{\Delta z} \ln^* \frac{cz_+ - a}{cz_- - a}, \tag{29}$$

where $\ln^* r = \ln|r| + \mathrm{i}(\arg r + 2n\pi)$ for $-\pi < \arg r \leqslant \pi$. Herein, n is a proper integer standing for the solution branch of the complex plane.

To show an application of the present method, we consider $Z = \ln \mathrm{e}^X \mathrm{e}^Y$ when $[X, Y] = \gamma Y$ holds for real γ. Now for the B_is of equation (15), it is direct that $x = (\gamma/2, 0, 0)$ and $y = (0, 1, 0)$. In this case, the present method is not directly applicable because it is based on the two different solutions of the quadratic equation, $f_v(\rho) = 0$ for $v = x$, y, or z (see equations (16) and (22)). Thus instead one may introduce a dummy variable to observe the procedure of the present method, and then take zero limit of it later. For the present example, since $x_2 = y_1 = 0$ is the obstacle, $x_2 = y_1 = \epsilon$ is used first, and then the $\epsilon \to 0$ limit is considered later. This kind of manipulation is justified by the fact that equation (22) covers the solutions obtained when $f_v(\rho)$ is constant or linear with respect to ρ. After a little algebra, one obtains $a = -2\gamma$, $b = 0$, $c = 2\gamma\,\mathrm{e}^{\gamma}$, $d = -2\gamma\,\mathrm{e}^{\gamma}$ in equation (23), which gives $z_+ = (\mathrm{e}^{\gamma} - 1)/\mathrm{e}^{\gamma}$ and $z_- = 0$ for $\gamma \geqslant 0$ or $z_+ = 0$ and $z_- = (\mathrm{e}^{\gamma} - 1)/\mathrm{e}^{\gamma}$ for $\gamma < 0$. These $z_\pm$s lead to $z_2 = (\gamma + \mathrm{i}2n\pi)\,\mathrm{e}^{\gamma}/(\mathrm{e}^{\gamma} - 1)$ in common. Since the $k_{w_1,\ldots,w_n}$s in equation (3) as well as x and y are real, the z now in consideration becomes also real. To do so, z_2 in equation (29) has to be real, which is possible only for $n = 0$. Consequently, one finds $z = (\gamma/2, \gamma/(1 - \mathrm{e}^{-\gamma}), 0)$ by equation (26), and thus it finally reads that $Z = X + (\gamma/(1 - \mathrm{e}^{-\gamma}))Y$. The traditional procedures suggested in [15, 17] also reveal the same result.

5. Remarks and conclusion

In the previous example, we show a case where $n = 0$ is selected as the solution branch in equation (29). This is the direct consequence of the following conditions: (i) $(a - d)/c$, b/c, and z are real and (ii) $z_\pm$ is also real in the example. The leading one always holds for real x and y. When x and y are real, $\rho_x(\rho_y(\rho_0; 1); 1)$ is always real for any real ρ_0. This guarantees that a, b, c and d are real up to a common phase factor $\mathrm{e}^{\mathrm{i}\delta}$ [22], and thus it follows that $(a - d)/c$ and b/c are real. This again leads to the fact that z are real-valued, which is already argued in the previous example in a different way. Meanwhile, the latter condition is given by the computation based on the actual value of x and y in the example. Although one may find the solution branch in this way, it is merely a problem-specific finding. Hence we remark that a systematic criterion to select solution branch in equation (29) is not established yet. The issue of making the present fixed-point method complete remains for future work.

We propose a procedure to figure out $\ln \mathrm{e}^X \mathrm{e}^Y$, the BCH solution, when X and Y are the elements of the split three-dimensional simple algebra (see equation (13) for the definition of this algebra). The procedure, named the fixed-point method, converts the manipulation of BCH formula into that of a differential equation. Therein the fixed point of the differential equation leads to the solution we are looking for. It is remarked that the validity of the fixed-point method is restricted to the case when the branch of the complex plane can be determined. Thus the issue regarding the selection of the solution branch remains for future work.

Acknowledgements

The author would like to acknowledge the encouragement of Professor M Howard Lee at the University of Georgia.

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